

# THE GENERATING FUNCTION OF THE EMBEDDING CAPACITY FOR 4-DIMENSIONAL SYMPLECTIC ELLIPSOIDS

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## Abstract

Quite recently, McDuff showed that the existence of a symplectic embedding of one four-dimensional ellipsoid into another can be established by comparing their corresponding sequences of ECH capacities. In this note we show that these sequences can be encoded in a generating function, which gives several new equivalent formulations of McDuff's theorem.

### 1. Embedding 4-dimensional Symplectic Ellipsoids.

We consider ellipsoids

$$E(a, b) := \left\{ z \in \mathbb{C}^2 : \frac{|z_1|^2}{a} + \frac{|z_2|^2}{b} \leq 1 \right\}$$

equipped with the standard symplectic structure  $\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$  of Euclidean space  $\mathbb{R}^4$ . The embedding problem in symplectic geometry asks if for given integers  $a, b, c, d > 0$  there exists a symplectic embedding  $\text{int } E(a, b) \xrightarrow{s} E(c, d)$ . Since each such embedding preserves the volume, an immediate obstruction for existence is  $ab \leq cd$ .

There are further obstructions which have their origin in embedded contact homology. Namely, define  $\mathcal{N}(a, b)$  to be the sequence of numbers from the set

$$\mathcal{S}(a, b) := \{ka + lb : k, l \in \mathbb{Z} \text{ and } k, l \geq 0\}$$

arranged in nondecreasing order with repetitions. For example, we have

$$\mathcal{N}(2, 3) = (0, 2, 3, 4, 5, 6, 6, 7, 8, 8, 9, 9, \dots).$$

For sequences of numbers  $\mathcal{A}$  and  $\mathcal{B}$  define a partial ordering by saying  $\mathcal{A} \preceq \mathcal{B}$  if, for all  $n \geq 0$ , the  $n$ -th entry of  $\mathcal{A}$  is not larger than the  $n$ -th entry of  $\mathcal{B}$ . Hutchings showed in [9] that an obstruction for the embedding problem is given by  $\mathcal{N}(a, b) \preceq \mathcal{N}(c, d)$ . Indeed, as conjectured by Hofer and recently proved by McDuff in [12], this is the only obstruction.

**THEOREM 1.** *There is a symplectic embedding  $\text{int } E(a, b) \xrightarrow{s} E(c, d)$  if and only if*

$$\mathcal{N}(a, b) \preceq \mathcal{N}(c, d).$$

Hence the embedding problem for symplectic ellipsoids can be reduced to studying the sequences  $\mathcal{N}(a, b)$ . Define a new sequence  $\mathcal{L}(a, b)$  by

$$\mathcal{L}_n(a, b) := \max\{j : \mathcal{N}_j(a, b) \leq n\} = \#\{m \in \mathcal{S}(a, b) : m \leq n\}.$$

From the definition it is clear, that

$$\mathcal{L}(a, b) \succeq \mathcal{L}(c, d) \iff \mathcal{N}(a, b) \preceq \mathcal{N}(c, d). \quad (1.1)$$

Geometrically,  $\mathcal{L}_n(a, b)$  corresponds to the number of lattice points in the triangle  $T_{a,b}^n$  bounded by  $x = 0$ ,  $y = 0$  and  $ax + by = n$ , including points on its boundary (Figure 1).

The aim of this note is to remark that the generating function of  $\mathcal{L}(a, b)$  is given by a surprisingly simple formula.

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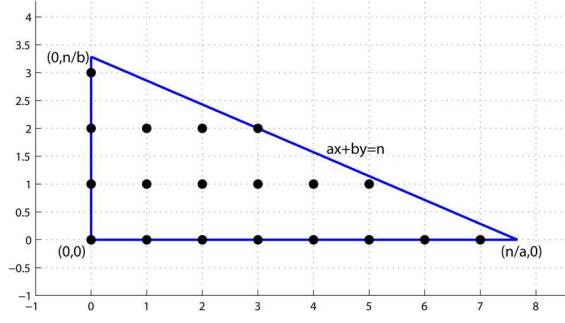


Figure 1: Interpreting  $\mathcal{L}_n(a, b)$  as a lattice count.

PROPOSITION 1. For  $0 \leq z < 1$  we have the expansion

$$\frac{1}{(1-z)(1-z^a)(1-z^b)} = \sum_{n=0}^{\infty} \mathcal{L}_n(a, b) z^n. \quad (1.2)$$

PROOF: We have

$$\begin{aligned} \frac{1}{(1-z)(1-z^a)(1-z^b)} &= \left( \sum_{k=0}^{\infty} z^k \right) \left( \sum_{l=0}^{\infty} z^{al} \right) \left( \sum_{m=0}^{\infty} z^{bm} \right) \\ &= \sum_{n=0}^{\infty} \left( \# \{ (k, l, m) \in \mathbb{Z}^3 : k, l, m \geq 0 \text{ and } k + al + bm = n \} \right) z^n \\ &= \sum_{n=0}^{\infty} \left( \# \{ (l, m) \in \mathbb{Z}^2 : l, m \geq 0 \text{ and } al + bm \leq n \} \right) z^n = \sum_{n=0}^{\infty} \mathcal{L}_n(a, b) z^n. \end{aligned}$$

□

There is also a geometric interpretation behind this formula, which will be explained in the next section. Note that  $\mathcal{L}_n(a, b)$  corresponds to the number of partitions of  $n$  into parts of size 1,  $a$  or  $b$  which is known as a denumerant problem. In this case one always obtains a rational generating function with poles that are roots of unity. Multiplying both sides of (1.2) by the denominator and comparing coefficients leads to the linear recurrence relation

$$\begin{aligned} \mathcal{L}_n(a, b) &= \mathcal{L}_{n-1}(a, b) + \mathcal{L}_{n-a}(a, b) + \mathcal{L}_{n-b}(a, b) + \mathcal{L}_{n-a-b-1}(a, b) \\ &\quad - \mathcal{L}_{n-a-1}(a, b) - \mathcal{L}_{n-b-1}(a, b) - \mathcal{L}_{n-a-b}(a, b) \end{aligned}$$

for  $n > 0$ . To initiate we take  $\mathcal{L}_0(a, b) = 1$  and set  $\mathcal{L}_n(a, b) := 0$  for  $n < 0$ . The following relation can be proved in an elementary way (see [6], section 5.6).

PROPOSITION 2. For  $n > 0$  we have

$$\mathcal{L}_n(a, b) = \mathcal{L}_{n-1}(a, b) + \left\lfloor \frac{n}{ab} \right\rfloor + \varepsilon(n) \quad (1.3)$$

where  $\varepsilon(n)$  is either 0 or 1 and its value just depends on the remainder

$$[n] \in \frac{\mathbb{Z}}{ab\mathbb{Z}}.$$

In some sense the whole information of  $\mathcal{L}(a, b)$  is therefore stored in its first  $ab$  terms. Moreover, one obtains the asymptotic behaviour

$$\mathcal{L}_n(a, b) \sim \frac{n^2}{2ab}.$$

In the following, we denote the generating function by

$$g_{a,b}(z) = \frac{1}{(1-z)(1-z^a)(1-z^b)}.$$

Denote further by  $f^{(k)}$  the  $k$ -th derivative of a function  $f$ . Via Cauchy's integral formula we compute

$$\mathcal{L}_n(a,b) = \frac{g_{a,b}^{(n)}(0)}{n!} = \frac{1}{2\pi i} \int_{\gamma} \frac{g_{a,b}(\xi) d\xi}{\xi^{n+1}} = \frac{1}{2\pi i} \int_{\gamma} \frac{d\xi}{(1-\xi)(1-\xi^a)(1-\xi^b)\xi^{n+1}},$$

which might be useful for numerical purposes.

On the space  $\mathcal{C}^\infty((-1,1), \mathbb{R})$  consider the partial ordering by saying  $f \preceq g$  iff  $f^{(k)}(x) \leq g^{(k)}(x)$  for all  $k \geq 0$  and  $x \in [0,1)$ . Putting things together we obtain the following

**COROLLARY 1.** *There is a symplectic embedding  $\text{int } E(a,b) \xrightarrow{s} E(c,d)$  if and only if one of the following equivalent conditions is fulfilled:*

- (a)  $\mathcal{N}(a,b) \preceq \mathcal{N}(c,d)$
- (b)  $\mathcal{L}(a,b) \succeq \mathcal{L}(c,d)$
- (c)  $g_{a,b} \succeq g_{c,d}$

**PROOF:** The equivalence of (a) and (b) was already noticed in (1.1). Now (b) implies for any integer  $k \geq 0$  and  $z \in [0,1)$

$$g_{a,b}^{(k)}(z) = \sum_{n=k}^{\infty} k! \binom{n}{k} \mathcal{L}_n(a,b) z^{n-k} \geq \sum_{n=k}^{\infty} k! \binom{n}{k} \mathcal{L}_n(c,d) z^{n-k} = g_{c,d}^{(k)}(z).$$

On the other hand (c) leads to

$$\mathcal{L}_k(a,b) = \frac{g_{a,b}^{(k)}(0)}{k!} \geq \frac{g_{c,d}^{(k)}(0)}{k!} = \mathcal{L}_k(c,d).$$

□

Thus the embedding question  $\text{int } E(a,b) \xrightarrow{s} E(c,d)$  relates to the problem if all coefficients of

$$G_{a,b,c,d}(z) := \frac{(1-z^c)(1-z^d) - (1-z^a)(1-z^b)}{(1-z)(1-z^a)(1-z^b)(1-z^c)(1-z^d)} = g_{a,b}(z) - g_{c,d}(z) = \sum_{n=0}^{\infty} (\mathcal{L}_n(a,b) - \mathcal{L}_n(c,d)) z^n$$

are nonnegative. Since  $G_{a,b,c,d}$  is again a rational function, its coefficients satisfy a linear recurrence. In [4], Conjecture 2 it is conjectured that each rational function, whose dominating poles (i.e. the ones of maximal modulus) do not lie on  $\mathbb{R}_+$ , has infinitely many positive and infinitely many negative coefficients in its power series expansion. Of course, we cannot apply this to  $G_{a,b,c,d}$ , since all of its poles have modulus 1 and  $1 \in \mathbb{R}_+$  occurs among them. One of the most celebrated results in the theory of linear recurrence sequences is the Skolem-Mahler-Lech theorem. It asserts that if a sequence  $(a_n)$  satisfies a linear recurrence relation, then the zero set

$$\{n \in \mathbb{N} : a_n = 0\}$$

is the union of a finite set and finitely many arithmetic progressions.

Let us use the approach via generating functions to check algebraically that for each positive integer  $n \in \mathbb{N}$  there is a symplectic embedding

$$\text{int } E(1, n^2) \xrightarrow{s} B(n).$$

Here the latter denotes the ball  $B(n) := E(n,n)$  of radius  $n$ . Geometrically, this corresponds to a filling of  $B(n)$  by  $n^2$  equal symplectic balls (Proposition 2.2 in [10]). The possibility of such a filling can be quite easily observed via toric models. For details we refer the reader to the survey paper [10].

With the lattice count interpretation we have

$$\mathcal{L}_k(n, n) = d\left(\left\lfloor \frac{k}{n} \right\rfloor\right),$$

where  $d(k) := \frac{1}{2}(k+1)(k+2)$  denotes the  $k$ -th triangle number. Consequently, by Proposition 1

$$g_{n,n}(z) = \frac{1}{(1-z)(1-z^n)^2} = \sum_{k=0}^{\infty} d\left(\left\lfloor \frac{k}{n} \right\rfloor\right) z^k.$$

For integers  $k \geq 0$  set

$$c(k) = \begin{cases} 1 & \text{if } k \equiv 0 \pmod{n}, \\ -1 & \text{if } k \equiv 1 \pmod{n}, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \frac{(1-z^n)^2}{(1-z)(1-z^{n^2})} &= \frac{1-z^n}{1-z} \cdot (1-z^n) \sum_{k=0}^{\infty} z^{kn^2} = (1+z+\dots+z^{n-1}) \sum_{k=0}^{\infty} (z^{kn^2} - z^{(kn+1)n}) \\ &= \sum_{k=0}^{\infty} c\left(\left\lfloor \frac{k}{n} \right\rfloor\right) z^k, \end{aligned}$$

such that

$$g_{1,n^2}(z) = \frac{g_{1,n^2}(z)}{g_{n,n}(z)} \cdot g_{n,n}(z) = \frac{(1-z^n)^2}{(1-z)(1-z^{n^2})} \cdot g_{n,n}(z) = \left( \sum_{k=0}^{\infty} c\left(\left\lfloor \frac{k}{n} \right\rfloor\right) z^k \right) \left( \sum_{l=0}^{\infty} d\left(\left\lfloor \frac{l}{n} \right\rfloor\right) z^l \right).$$

In view of (1.2) it suffices to show for each nonnegative integer  $N$

$$\sum_{k=0}^N c\left(\left\lfloor \frac{k}{n} \right\rfloor\right) d\left(\left\lfloor \frac{N-k}{n} \right\rfloor\right) \geq d\left(\left\lfloor \frac{N}{n} \right\rfloor\right). \quad (1.4)$$

For given  $N \geq 0$  we pick integers  $0 \leq p, q, r$  with  $q, r < n$  such that  $N = pn^2 + qn + r$ . Setting  $d(-1) = d(-2) := 0$ , we obtain from the periodicity of  $c(k)$

$$\begin{aligned} \sum_{k=0}^N c\left(\left\lfloor \frac{k}{n} \right\rfloor\right) d\left(\left\lfloor \frac{N-k}{n} \right\rfloor\right) &= \sum_{j=0}^p ((r+1)d(jn+q) + (n-r-1)d(jn+q-1)) \\ &\quad - \sum_{j=0}^p ((r+1)d(jn+q-1) + (n-r-1)d(jn+q-2)) \\ &= \sum_{j=0}^p ((r+1)(jn+q+1) + (n-r-1)(jn+q)) \\ &= \frac{p(p+1)}{2}n^2 + (p+1)qn + (p+1)(r+1) = (p+1)(N+1) - \frac{p(p+1)}{2}n^2. \end{aligned}$$

For  $q < n$ ,  $n \geq 2$  we have

$$\frac{3q}{2} + \frac{q^2}{2} = \frac{q(q+1)}{2} + q \leq \frac{nq}{2} + \frac{nq}{2},$$

such that  $\frac{3q}{2} + \frac{q^2}{2} \leq qn$  holds for all nonnegative integers  $q < n$ . One also easily checks that  $\frac{3pn}{2} \leq \frac{pn^2}{2} + p$  holds for all nonnegative integers  $p, n$ . Thus

$$\begin{aligned} (p+1)(N+1) &\geq (p+1)(pn^2 + qn + 1) = p^2n^2 + pn^2 + pqn + qn + p + 1 \\ &\geq p^2n^2 + \frac{pn^2}{2} + \frac{3pn}{2} + pqn + \frac{q^2}{2} + \frac{3q}{2} + 1 = \frac{(pn+q+1)(pn+q+2)}{2} + \frac{p(p+1)}{2}n^2 \\ &= d\left(\left\lfloor \frac{N}{n} \right\rfloor\right) + \frac{p(p+1)}{2}n^2 \end{aligned}$$

shows that (1.4) is valid.

The symplectic capacity function  $c : [1, \infty) \rightarrow \mathbb{R}$  defined by

$$c(a) := \inf \left\{ \mu : \text{int } E(1, a) \xrightarrow{s} B(\mu) \right\}$$

is studied in detail in [11]. We just computed  $c(a^2) = a$  for positive integers  $a$ . Indeed,  $c(a) = \sqrt{a}$  holds for  $a \in \mathbb{N}$  if  $a$  is 1, 4 or  $\geq 9$ . The other values for integral  $a$  are given by

$$c(2) = c(3) = 2, \quad c(5) = c(6) = \frac{5}{2}, \quad c(7) = \frac{8}{3}, \quad c(8) = \frac{17}{6}.$$

We finish this section by remarking that Theorem 1 does not hold in higher dimensions. Counterexamples are due to Guth [5] and Hind-Kerman [7]. Even worse, embedded contact homology only exists in dimension 4 and there is so far no good guess of what a criterion for embedding ellipsoids could be.

**2. Counting Lattice Points in Polyhedra.** Let  $P \subset \mathbb{R}^d$  be a polyhedron. In order to count the lattice points in  $P$  one associates the generating function

$$\sum_{m \in P \cap \mathbb{Z}^d} \mathbf{x}^m \quad \text{with } \mathbf{x}^m = x_1^{\mu_1} \cdots x_d^{\mu_d}$$

for  $m = (\mu_1, \dots, \mu_d)$ . The total number of lattice points in  $P$  is then given by the value of the generating function at  $\mathbf{x} = (1, \dots, 1)$ . The advantage of this approach is that these generating functions can still be computed for cones  $K \subset \mathbb{R}^d$ , which actually contain an infinite number of lattice points. A cone is characterized by the property that  $0 \in K$  and for every  $x \in K$  and  $\lambda \geq 0$  one has  $\lambda x \in K$ . For example, the generating function of the non-negative orthant is given by

$$\sum_{m \in \mathbb{R}_+^d \cap \mathbb{Z}^d} \mathbf{x}^m = \prod_{i=1}^d \frac{1}{1 - x_i}.$$

The generating function of a polyhedron  $P$  is calculated as the sum of generating functions of tangent cones at the vertices of  $P$ , for details see [2].

Usually a cone  $K$  is given as a span of vectors  $u_1, \dots, u_k \in \mathbb{R}^d$ ,

$$K = \text{co}(u_1, \dots, u_k),$$

meaning that every vector  $v \in K$  can be written as a sum  $v = \sum \lambda_i v_i$  with  $\lambda_i \geq 0$ . A cone  $K$  is called unimodular, if it is spanned by  $u_1, \dots, u_d \in \mathbb{Z}^d$  and these vectors form a basis of the lattice. Generating functions for unimodular cones are particularly easy to calculate. Unfortunately, all tangent cones of the triangle  $T_{a,b}^n$  are unimodular only if  $a = b$ . Hence we cannot expect an easy formula for  $a \neq b$ , also we have already seen that the number of lattice points in  $T_{a,a}^n$  is given by

$$d \left( \left\lfloor \frac{n}{a} \right\rfloor \right).$$

Instead consider  $T_{a,b}^n = \{x, y \in \mathbb{R}_+^2 : ax + by \leq n\}$  as lying in the hyperplane  $z \equiv n$  in  $\mathbb{R}^3$ . Then

$$\bigcup_{n \geq 0} T_{a,b}^n \cap \mathbb{Z}^3 = \text{co} \left( \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ b \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \cap \mathbb{Z}^3.$$

The latter cone is unimodular and has generating function

$$f(x, y, z) = \frac{1}{(1 - xz^a)(1 - yz^b)(1 - z)}.$$

In particular, the number of lattice points in  $T_{a,b}^n$  corresponds to the coefficient of  $z^n$  of the expansion of  $f$  restricted to  $x = y = 1$ . This explains formula (1.2).

**3. Scale Invariance.** The condition in Theorem 1 is scale invariant, meaning that for each real  $\lambda > 0$  one has

$$\mathcal{N}(a, b) \preceq \mathcal{N}(c, d) \iff \mathcal{N}(\lambda a, \lambda b) \preceq \mathcal{N}(\lambda c, \lambda d).$$

Unfortunately, this scale invariance does not descend to the generating functions. Thus  $g_{a,b} \succeq g_{c,d}$  does not imply  $g_{\lambda a, \lambda b} \succeq g_{\lambda c, \lambda d}$  and it does not make sense to extend our notion of generating functions to real parameters  $a, b$ . For rational  $a, b, c, d \in \mathbb{Q}_+$  the best one could do is to choose  $N \in \mathbb{N}$  such that  $Na, Nb, Nc, Nd$  are integers and then compare the generating functions  $g_{Na, Nb}$  and  $g_{Nc, Nd}$ .

The embedding condition  $g_{a,b} \succeq g_{c,d}$  requires

$$g_{a,b}(z) \geq g_{c,d}(z) \tag{3.1}$$

for all  $z \in [0, 1)$ . But (3.1) is scale invariant, since it is equivalent to

$$\frac{(1 - z^c)(1 - z^d)}{(1 - z^a)(1 - z^b)} \geq 1$$

and one may substitute  $z = w^\lambda$  with  $w \in [0, 1)$  on the left hand side. Therefore it corresponds to an embedding obstruction which extends to real parameters  $a, b$ . The following lemma shows that at least in the case of embeddings into a ball this obstruction is the volume constraint.

**LEMMA 1.** *Let  $a, b, c, d \in \mathbb{R}$  be positive, such that  $b \leq \min(c, d)$ . Then the inequality*

$$g_{a,b}(z) \geq g_{c,d}(z)$$

*holds for all  $z \in [0, 1)$  if and only if  $a$  is chosen such that  $ab \leq cd$ .*

**PROOF:** By scale invariance it suffices to show that under the assumption  $b \leq \min(1, c)$  the inequality

$$(1 - z)(1 - z^c) \geq (1 - z^a)(1 - z^b) \tag{3.2}$$

holds for all  $z \in (0, 1)$  if and only if  $a \leq \frac{c}{b}$ .

We first consider the case  $c = ab$ , such that  $b \leq 1 \leq a$ . Then we have

$$ab \leq \min(a, ab + 1) \leq \max(a, ab + 1) \leq a + b.$$

The function  $f(x) = z^x$  is convex and monotone decreasing for fixed  $z \in (0, 1)$  and  $x \in (0, \infty)$ . Hence the segment from  $(ab, z^{ab})$  to  $(a + b, z^{a+b})$  lies above the segment from  $(a, z^a)$  to  $(ab + 1, z^{ab+1})$ . Comparing the heights of intersection of these segments with the horizontal line  $x = \frac{b(ab+1)+a}{b+1}$  yields the estimate

$$\frac{b}{b+1}z^{ab+1} + \frac{1}{b+1}z^a \leq \frac{b}{b+1}z^{ab} + \frac{1}{b+1}z^{a+b}.$$

Considering the function  $F : [1, \infty) \rightarrow \mathbb{R}$ ,

$$F(a) = z^{ab+1} + z^a + z^b - z^{ab} - z^{a+b} - z$$

for fixed  $z \in (0, 1)$  and  $b \leq 1$ , the previous inequality implies that  $f$  is monotone increasing in  $a$ . Consequently,  $F(a) \geq F(1) = 0$ . This tells us that (3.2) holds for all  $z \in (0, 1)$  if  $c = ab$ . Since increasing  $c$  only increases the left hand side of (3.2), we have shown that this inequality is satisfied for all  $z \in (0, 1)$  if  $c \geq ab$ .

Now we fix any  $0 < \lambda < 1$  and consider the case  $c = \lambda ab$ . Let

$$C := \frac{\lambda^2 ab^2 + a + b}{\lambda b + 1} > \frac{1 + b}{\lambda b + 1} > 1.$$

Choose  $\delta > 0$  small enough, such that

$$z^{C-\lambda b} \geq -\frac{(a+b)^2}{4(1-\lambda)b} \log z$$

holds for  $z \in (1 - \delta, 1)$ . Using this and the convexity and monotonicity of the function  $f$ , we obtain for  $1 \leq \tau \leq a$

$$\begin{aligned} \frac{\lambda b}{\lambda b + 1} z^{\lambda \tau b + 1} + \frac{1}{\lambda b + 1} z^\tau &\geq f\left(\frac{\lambda^2 \tau b^2 + \tau + \lambda b}{\lambda b + 1}\right) \geq f\left(\frac{\lambda^2 \tau b^2 + \tau + b}{\lambda b + 1}\right) - \frac{(1 - \lambda)b}{\lambda b + 1} f'\left(\frac{\lambda^2 \tau b^2 + \tau + b}{\lambda b + 1}\right) \\ &> f\left(\frac{\lambda^2 \tau b^2 + \tau + b}{\lambda b + 1}\right) - \frac{(1 - \lambda)b}{2} f'(C) = f\left(\frac{\lambda^2 \tau b^2 + \tau + b}{\lambda b + 1}\right) - \frac{(1 - \lambda)b}{2} z^C \log z \\ &\geq f\left(\frac{\lambda^2 \tau b^2 + \tau + b}{\lambda b + 1}\right) + \frac{(a + b)^2}{8} z^{\lambda b} (\log z)^2 = f\left(\frac{\lambda^2 \tau b^2 + \tau + b}{\lambda b + 1}\right) + \frac{(a + b)^2}{8} f''(\lambda b). \end{aligned}$$

We now apply the inequality

$$|\mu f(x) + (1 - \mu)f(y) - f(\mu x + (1 - \mu)y)| \leq \frac{|x - y|^2}{8} \cdot \max_{\xi \in [x, y]} f''(\xi)$$

with  $\mu = \frac{\lambda b}{\lambda b + 1}$  to conclude

$$\begin{aligned} \frac{\lambda b}{\lambda b + 1} z^{\lambda \tau b + 1} + \frac{1}{\lambda b + 1} z^\tau &> f(\mu(\lambda \tau b) + (1 - \mu)(\tau + b)) + \frac{|\lambda \tau b - (\tau + b)|^2}{8} \cdot \max_{\xi \in [\lambda \tau b, \tau + b]} f''(\xi) \\ &\geq \mu f(\lambda \tau b) + (1 - \mu)f(\tau + b) = \frac{\lambda b}{\lambda b + 1} z^{\lambda \tau b} + \frac{1}{\lambda b + 1} z^{\tau + b} \end{aligned}$$

for  $1 \leq \tau \leq a$  and  $z \in (1 - \delta, 1)$ . Consequently, the function  $F_\lambda : [1, a] \rightarrow \mathbb{R}$  defined by

$$F_\lambda(\tau) = z^{\lambda \tau b + 1} + z^\tau + z^b - z^{\lambda \tau b} - z^{\tau + b} - z$$

is monotone decreasing for  $z \in (1 - \delta, 1)$ . Hence for these values of  $z$  we have

$$F_\lambda(\tau) \leq F_\lambda(1) = (1 - z)(z^b - z^{\lambda b}) < 0.$$

This shows that (3.2) is violated for  $c = \lambda ab$  with  $0 < \lambda < 1$ . □

**Acknowledgements.** I warmly thank Dusa McDuff for introducing me into the topic at Edifest 2010 and giving helpful comments. I also thank Felix Schlenk and Matthias Schwarz for making it possible for me to participate at this conference. Finally, I thank the Max Planck Institute for Mathematics in the Sciences for support and providing a pleasant environment to do this research.

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